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How to embed an arbitrary Hamiltonian dynamics in a superintegrable (or just integrable) Hamiltonian dynamics

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Abstract

Given an *arbitrary* (autonomous) Hamiltonian $h(\vec{p};\vec{q})$, where the N components p_n of the N-vector $\vec{p}$ are the canonical momenta, the N components q_n of the N-vector $\vec{q}$ are the corresponding canonical coordinates and N is an *arbitrary* positive integer, we show how to manufacture (autonomous) Hamiltonians $\tilde{H}(\vec{\tilde{p}},\tilde{P};\vec{\tilde{q}},\tilde{Q})$, featuring the $N+1$ canonical momenta $\tilde{p}_n$, $\tilde{P}$ and the corresponding canonical coordinates $\tilde{q}_n$, $\tilde{Q}$, and having the following two properties: (i) the *generic* solutions of $\tilde{H}(\vec{\tilde{p}},\tilde{P};\vec{\tilde{q}},\tilde{Q})$ are *periodic*—entailing that the dynamics yielded by this Hamiltonian $\tilde{H}(\vec{\tilde{p}},\tilde{P};\vec{\tilde{q}},\tilde{Q})$ is (maximally) *superintegrable*, namely, it features $2N+1$ functionally independent constants of motion, $N+1$ of which in involution. (ii) On the manifold characterized by the condition $\tilde{H}(\vec{\tilde{p}},\tilde{P};\vec{\tilde{q}},\tilde{Q})=0$, the coordinates $\tilde{P}$ and $\tilde{Q}$ evolve trivially, $\tilde{P}(t)=\tilde{P}(0)$ and $\tilde{Q}(t)=\tilde{Q}(0)+t$, while the evolution of the $2N$ coordinates $\tilde{p}_n(t),\tilde{q}_n(t)$ is that determined by the (*arbitrary*!) Hamiltonian $h(\vec{\tilde{p}};\vec{\tilde{q}})$. This is related to an earlier finding by Bolsinov and Taimanov.

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1. Introduction

Almost a decade ago Bolsinov and Taimanov [1, 2] presented, in a geometrical setting, a remarkable finding, the description of which can be reported in the simple language of dynamical systems as follows: it is the example of a Hamiltonian dynamical system the *generic* time-evolution of which is *integrable*, except when it is restricted to some special invariant manifold of the system, on which it is instead *nonintegrable*. More specifically, the motion on that special manifold is equivalent to a torus automorphism, which is an Anosov system and therefore has strongly chaotic properties. This finding was unexpected and perhaps it was considered surprising by many—although of course it did not violate any established notion. It entailed the possibility to *embed* a *nonintegrable* dynamics inside an *integrable* one—of course with a proper significance of the term 'to embed', as loosely entailed by the previous discussion and precisely defined below.

On the other hand quite recently, in the context of Hamiltonian dynamics, we have shown how to extend an *arbitrary* (autonomous) Hamiltonian so that the (also autonomous) Hamiltonian thereby obtained, which generally features very few additional variables, exhibits the following two properties: (i) *all* the orbits yielded by its dynamics are *periodic* with *arbitrarily* assigned periods (possibly all equal, so that the evolution is *isochronous*), entailing that this extended Hamiltonian is *maximally superintegrable*, namely, it has the maximal number of (functionally independent) conserved quantities; (ii) when restricted to the original variables, the generic dynamics yielded by the extended Hamiltonian essentially coincides (up to a rescaling of the time variable) with that yielded by the original (unmodified) Hamiltonian, over time intervals which are *short* relative to the (*arbitrarily* assigned) periodicity of the extended Hamiltonian [3–6]. When the result by Bolsinov and Taimanov outlined above was brought to our attention, we realized that, via an approach similar to that underlying our findings as just described, it is possible to provide an analogous, but considerably more general, result. Its essence is conveyed by the abstract of this paper (and see also remark 3). A precise formulation of it, and its proof, is provided in the following section. An explicit example—with the original, arbitrary Hamiltonian being itself *integrable* but generally *not superintegrable*—is provided in the last section. Simple as this example is, it sheds light on a natural question evoked by our finding, namely, how some of the constants of motion of the extended *superintegrable* system disappear when its evolution is restricted to the invariant manifold on which its dynamics is instead *nonintegrable* (or at least *less integrable*).

Let us end this introductory section by emphasizing that the possibility to embed an *arbitrary* Hamiltonian dynamics in a *superintegrable* (or just *integrable*) Hamiltonian dynamics should not be expected to simplify the treatment of the dynamics entailed by an *arbitrary* Hamiltonian; although it is conceivable that in some case it might be helpful to shed some light on some of its properties—as illustrated by the simple example treated below.

2. Results

Theorem. *Let the Hamiltonian $H(P;Q)$, depending on the two canonical variables P and Q, have the following two properties: (i) for $P(0)=0$, this Hamiltonian yields the simple solution*

$$P(t)=0, \qquad Q(t)=Q(0)+t, \tag{1a}$$

entailing that the relation

$$P=0 \tag{1b}$$

identifies an invariant manifold of this Hamiltonian; (ii) for $P(0)\neq 0$, the evolution of the canonical variables is instead periodic, namely, there exists a nonnegative period T—depending on the initial data $P(0)$ and $Q(0)$—such that, for all time t,

$$P(t+T)=P(t), \qquad Q(t+T)=Q(t). \tag{2}$$

Note that such Hamiltonians certainly exist; an example is exhibited below.

Let $h(\vec{p};\vec{q})$ be another Hamiltonian, having as canonical momenta the N components p_n of the N-vector $\vec{p}$ and as canonical coordinates the N components of the N-vector $\vec{q}$, and such that the time evolution of these canonical variables exists globally (for all time); aside for this mild restriction (and possibly additional restrictions guaranteeing the smoothness of the time-dependence of this evolution, see below), this Hamiltonian $h(\vec{p};\vec{q})$ is arbitrary (with N an arbitrary positive integer). Note that, here and hereafter, the index n runs from 1 to N.

Finally let us introduce a third Hamiltonian $\tilde{H}(\vec{\tilde{p}},\tilde{P};\vec{\tilde{q}},\tilde{Q})$, having the $N+1$ momenta $\tilde{p}_n$, $\tilde{P}$ and the corresponding $N+1$ coordinates $\tilde{q}_n$, $\tilde{Q}$ as its canonical variables, and being defined as follows in terms of the two Hamiltonians $H(P,Q)$ and $h(\vec{\tilde{p}};\vec{\tilde{q}})$ introduced above:

$$\tilde{H}(\vec{\tilde{p}},\tilde{P};\vec{\tilde{q}},\tilde{Q}) = H(\tilde{P}+h(\vec{\tilde{p}};\vec{\tilde{q}});\tilde{Q}). \tag{3}$$

Here and hereafter the superimposed tilde is a reminder that the time evolution of the corresponding quantities is determined by this Hamiltonian $\tilde{H}$.

Note that this definition of the Hamiltonian $\tilde{H}$ implies that

$$c = h(\vec{\tilde{p}};\vec{\tilde{q}}) \tag{4}$$

is a constant of motion for the evolution it entails, hence the relation

$$\tilde{P}+c = \tilde{P}+h(\vec{\tilde{p}};\vec{\tilde{q}}) = 0 \tag{5}$$

identifies an invariant manifold for this Hamiltonian $\tilde{H}(\vec{\tilde{p}},\tilde{P};\vec{\tilde{q}},\tilde{Q})$ (see (1b) and (3)).

Then for any initial data off this invariant manifold, namely such that $\tilde{P}(0)+h(\vec{\tilde{p}}(0);\vec{\tilde{q}}(0)) \neq 0$, the time evolution yielded by this Hamiltonian, (3), is completely periodic with a period $\tilde{T}$ depending only on the two initial data $\tilde{P}(0)+h(\vec{\tilde{p}}(0);\vec{\tilde{q}}(0))$ and $\tilde{Q}(0)$,

$$\tilde{P}(t+\tilde{T}) = \tilde{P}(t), \qquad \tilde{Q}(t+\tilde{T}) = \tilde{Q}(t), \tag{6a}$$

$$\tilde{p}_n(t+\tilde{T}) = \tilde{p}_n(t), \qquad \tilde{q}_n(t+\tilde{T}) = \tilde{q}_n(t); \tag{6b}$$

while for initial data on the invariant manifold (5)—i.e., for arbitrary initial data $\vec{\tilde{p}}(0)$ and $\vec{\tilde{q}}(0)$ with $\tilde{P}(0)$ assigned, $\tilde{P}(0) = -h(\vec{\tilde{p}}(0);\vec{\tilde{q}}(0))$—the time evolution of the $2N$ variables $\tilde{p}_n,\tilde{q}_n$ is exactly the same as that yielded by the (arbitrary!) Hamiltonian h, i.e. by the equations of motion

$$\dot{\tilde{p}}_n = -\frac{\partial h(\vec{\tilde{p}};\vec{\tilde{q}})}{\partial \tilde{q}_n}, \qquad \dot{\tilde{q}}_n = \frac{\partial h(\vec{\tilde{p}};\vec{\tilde{q}})}{\partial \tilde{p}_n}. \tag{7}$$

This theorem, the proof of which is provided immediately below, details the precise meaning of the claim made in the title of this paper, in particular the significance of the term 'embed' used there.

Proof. The $2N+2$ equations of motion yielded by the Hamiltonian (3) read as follows:

$$\dot{\tilde{P}} = -\frac{\partial H(\tilde{P}+c;\tilde{Q})}{\partial \tilde{Q}}, \qquad \dot{\tilde{Q}} = \frac{\partial H(\tilde{P}+c;\tilde{Q})}{\partial \tilde{P}}, \tag{8a}$$

$$\dot{\tilde{p}}_n = -\frac{\partial H(\tilde{P}+c;\tilde{Q})}{\partial \tilde{P}}\frac{\partial h(\vec{\tilde{p}};\vec{\tilde{q}})}{\partial \tilde{q}_n}, \qquad \dot{\tilde{q}}_n = \frac{\partial H(\tilde{P}+c;\tilde{Q})}{\partial \tilde{P}}\frac{\partial h(\vec{\tilde{p}};\vec{\tilde{q}})}{\partial \tilde{p}_n}. \tag{8b}$$

Hence, via the second equation (8a), the equations of motion (8b) can be rewritten as follows:

$$\dot{\tilde{p}}_n = -\dot{\tilde{Q}}\frac{\partial h(\vec{\tilde{p}};\vec{\tilde{q}})}{\partial \tilde{q}_n}, \qquad \dot{\tilde{q}}_n = \dot{\tilde{Q}}\frac{\partial h(\vec{\tilde{p}};\vec{\tilde{q}})}{\partial \tilde{p}_n}. \tag{9}$$

Hence they clearly entail

$$\tilde{p}_n(t) = p_n(\tilde{Q}(t)), \qquad \tilde{q}_n(t) = q_n(\tilde{Q}(t)). \tag{10}$$

Let us re-emphasize that, for notational convenience, we denote as $\tilde{p}_n(t), \tilde{q}_n(t)$ the dynamical variables whose time-evolution is determined by the Hamiltonian $\tilde{H}$, see (3)—i.e., $\tilde{p}_n(t)$ and $\tilde{q}_n(t)$ are the solutions of the Hamiltonian equations (8) or, equivalently, (9) with (8*a*)—while $p_n(t), q_n(t)$ denote the dynamical variables whose time-evolution is determined by the Hamiltonian $h(\vec{p};\vec{q})$ (i.e., $p_n(t)$ and $q_n(t)$ are the solutions of the Hamiltonian equations (7) with all tildes removed); and of course the evolution of $\tilde{Q}(t)$ is determined by the Hamiltonian $H(\tilde{P}+c, \tilde{Q})$, see (8*a*) with the constant of motion c given by (4), while the initial data $p_n(0), q_n(0)$ are determined in terms of the initial data $\tilde{p}_n(0), \tilde{q}_n(0)$ by the relations (10) at $t = 0$.

It is then plain that the assumptions made above on the dynamics yielded by the Hamiltonian H entail the validity of the theorem: indeed if $\tilde{Q}(t)$ is a *periodic* function of time, the relation (10) implies that all the variables $\tilde{p}_n(t), \tilde{q}_n(t)$ are also *periodic* (with the same period), while if the time evolution of $\tilde{Q}(t)$ is given by the second formula (1*a*), the relation (10) implies that the evolution of $\tilde{p}_n(t), \tilde{q}_n(t)$ coincides (up to an irrelevant constant translation of the time variable) with the evolution of $p_n(t), q_n(t)$. □

Remark 1. All the time evolutions yielded by the Hamiltonian $\tilde{H}$, see (3)—except for those restricted to the invariant manifold of codimension 1 characterized by the relation (5)—are *completely periodic*, see (6). Hence this Hamiltonian $\tilde{H}$ is *maximally superintegrable*, featuring $2N + 1$ functionally independent constants of motion. While we believe this implication to be well known, we did provide a proof of it in [4]. Note that, while that proof was given for *isochronous* systems, it is equally valid for systems yielding *completely periodic* evolutions—namely evolutions, such as those relevant here, in which *all* the degrees of freedom evolve periodically, entailing that the corresponding trajectories are closed—even if the periods associated with different trajectories are not equal, namely the motions are *periodic* but not *isochronous*. And let us recall that from the usual proof of the Darboux' theorem [7] it follows that, out of the $2N + 1$ constants of motion possessed by this system, $N + 1$ constants of motion can be taken in involution. Of course, in order to be able to define the Poisson bracket between the various constants of motion [4], we require that they be twice differentiable, a property which follows from mild smoothness assumptions on the Hamiltonian h, which we do not pursue further here.

Remark 2. As mentioned in the formulation of the theorem there are many Hamiltonians $H(P, Q)$ yielding evolutions that satisfy the requirements specified in the (first part of the) theorem. Clearly the requirement that $P = 0$ be an invariant manifold characterized by the simple evolution (1*a*) is satisfied by any Hamiltonian such that

$$\text{for} \quad P = 0, \qquad \frac{\partial H(P,Q)}{\partial P} = 1, \qquad \frac{\partial H(P,Q)}{\partial Q} = 0; \tag{11}$$

while the requirement that, for $P \neq 0$, all the trajectories yielded by this Hamiltonian be closed can be enforced by making sure that all constant-energy contours of $H(P, Q)$ are bounded.

A specific Hamiltonian $H(P, Q)$ having these properties reads—of course, consistently with the conditions (11)—as follows:

$$H(P,Q) = P - P^2(Q^2+1). \tag{12a}$$

Hereafter, we use, for the constant value of the Hamiltonian on the trajectory under consideration, the notation

$$H(P,Q) = \frac{\sin^2\theta}{4}, \tag{12b}$$

where θ runs from 0 to 2π. (The quadrant to which θ belongs will be determined below). We thereby restrict attention to values of the Hamiltonian in the interval $0 \leqslant H(P, Q) \leqslant 1/4$. It is moreover easily seen that for $H(P, Q) = \theta = 0$ one gets just the evolution (1*a*), while for $H(P, Q) = 1/4, \theta = \pi/2$ or $\theta = 3\pi/2$ one gets the equilibrium configuration $P = 1/2, Q = 0$ (see also below).

The equation of motion for $Q(t)$ yielded by this Hamiltonian (12*a*) reads

$$\dot{Q} = 1 - 2P(Q^2 + 1), \tag{13a}$$

namely, via (12*a*) and (12*b*),

$$\dot{Q} = \sqrt{1 - (Q^2 + 1)\sin^2\theta} \tag{13b}$$

yielding the solution

$$Q(t) = \cot\theta \sin[(t + t_0)\sin\theta], \tag{14a}$$

and correspondingly, via (12),

$$P(t) = \frac{1 - \cos\theta\cos[(t + t_0)\sin\theta]}{2\{1 + \cot^2\theta\sin^2[(t + t_0)\sin\theta]\}}. \tag{14b}$$

These formulae contain two constants, θ and t_0, hence they provide the *general* solution of the equations of motion yielded by the Hamiltonian (12*a*). In the *initial-value* problem, the constant angle θ (including the quadrant it belongs to) and the constant t_0 (up to an irrelevant $\mathrm{mod}(2\pi/\sin\theta)$ ambiguity) are determined in terms of the initial data $Q(0)$ and $P(0)$ by the requirement that these formulae, (14), hold at $t = 0$. And of course the *periodic* character of this solution is plain.

Remark 3. It is, of course, possible to modify the Hamiltonian $\tilde{H}$ (say, by adding to it $\tilde{H}^2$ times a function of the other constants of motion) so that the resulting dynamics becomes *integrable* rather than *superintegrable*, but does not change at all on the invariant manifold $\tilde{H} = \tilde{P} = 0$. This justifies the parenthetical insert in the title of this paper.

3. An example

We complete this paper by reporting a simple example, characterized by the Hamiltonian $H(P, Q)$ (12*a*), and by the assignment $N = 2$ with the rather trivial Hamiltonian

$$h(p_1, p_2; q_1, q_2) = \frac{1}{2}\sum_{n=1}^{2}\left(p_n^2 + \omega_n^2 q_n^2\right). \tag{15}$$

The explicit solution of this Hamiltonian reads of course as follows:

$$q_n(t) = q(0)\cos(\omega_n t) + p_n(0)\frac{\sin(\omega_n t)}{\omega_n}, \tag{16a}$$

$$p_n(t) = p(0)\cos(\omega_n t) - q_n(0)\omega_n\sin(\omega_n t). \tag{16b}$$

Here and hereafter the index n runs from 1 to 2, for definiteness we assume the two constants ω_1 and ω_2 to be positive, $\omega_1 > 0, \omega_2 > 0$, and we will denote with α their ratio, $\alpha = \omega_1/\omega_2$. Note that these formulae entail the relations

$$p_n(t) \pm \mathrm{i}\omega_n q_n(t) = [p_n(0) \pm \mathrm{i}\omega_n q_n(0)]\exp(\pm\mathrm{i}\omega_n t). \tag{16c}$$

This Hamiltonian h, see (15), is of course integrable, since the two quantities

$$c_n = p_n^2 + \omega_n^2 q_n^2 \tag{17}$$

provide obviously two constants of motion. If and only if the constant α is a *rational* number,

$$\alpha = \frac{\omega_1}{\omega_2} = \frac{r}{s}, \tag{18}$$

with r and s, here and hereafter, two positive coprime integers, this system is moreover *superintegrable*, since then the quantity

$$C = [p_1(t) + \mathrm{i}\omega_1 q_1(t)]^r \, [p_2(t) - \mathrm{i}\omega_2 q_2(t)]^s \,, \tag{19}$$

which is obviously time-independent (see (16*c*) and (18)), provides a third *one-valued globally defined* constant of motion. In fact the real and imaginary parts of this quantity—a polynomial in the four dependent variables p_1, p_2, q_1, q_2—provide two constants of motion, but obviously only three out of the four real constants of motion $c_1, c_2, \mathrm{Re}\,[C]\,, \mathrm{Im}\,[C]$ are functionally independent, indeed clearly

$$|C|^2 = (\mathrm{Re}[C])^2 + (\mathrm{Im}[C])^2 = c_1^r c_2^s, \tag{20}$$

see (19) and (17).

Let us now consider the Hamiltonian $\tilde{H}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q})$ defined by (3) with (15). As implied by the treatment provided in the preceding section (see (14) and (10)) the solution of its equations of motion reads as follows:

$$\tilde{Q}(t) = \cot\tilde{\theta}\,\sin[(t+t_0)\sin\tilde{\theta}], \tag{21a}$$

$$\tilde{P}(t) = \frac{1-\cos\tilde{\theta}\cos[(t+t_0)\sin\tilde{\theta}]}{2\{1+\cot^2\tilde{\theta}\,\sin^2[(t+t_0)\sin\tilde{\theta}]\}}, \tag{21b}$$

$$\tilde{q}_n(t) = \tilde{q}(0)\cos[\omega_n\tilde{Q}(t)] + \tilde{p}_n(0)\frac{\sin[\omega_n\tilde{Q}(t)]}{\omega_n}, \tag{21c}$$

$$\tilde{p}_n(t) = \tilde{p}(0)\cos[\omega_n\tilde{Q}(t)] - \tilde{q}_n(0)\omega_n\sin[\omega_n\tilde{Q}(t)], \tag{21d}$$

with the constant $\tilde{\theta}$ related, to the Hamiltonian $\tilde{H}(\vec{\tilde{p}}, \tilde{P}; \vec{\tilde{q}}, \tilde{Q})$ defined by (3) with (15), by the formula (see (12*b*))

$$\sin^2\tilde{\theta}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q}) = 4\tilde{H}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q}). \tag{22}$$

Note that—to get a more explicit expression of the solution—the variable $\tilde{Q}(t)$ in (21*c*) and (21*d*) should be replaced by its expression (21*a*); and that thereby these solution formulae, (21), imply the relations

$$\tilde{p}_n(t) \pm \mathrm{i}\omega_n\tilde{q}_n(t) = [\tilde{p}_n(0) \pm \mathrm{i}\omega_n\tilde{q}_n(0)]\exp\{\pm\mathrm{i}\omega_n\cot\tilde{\theta}\,\sin[(t+t_0)\sin\tilde{\theta}]\}. \tag{23}$$

We know from the results of the preceding section that the system characterized by this Hamiltonian $\tilde{H}$ is *superintegrable*, featuring five globally-defined one-valued functionally-independent constants of motion: indeed an explicit definition of five such constants of motion is provided by the Hamiltonian $\tilde{H}$ itself and by the real and imaginary parts of the two complex quantities

$$\tilde{C}_n(\tilde{p}_n; \tilde{q}_n, \tilde{Q}) = (\tilde{p}_n + \mathrm{i}\omega_n\tilde{q}_n)\exp(-\mathrm{i}\omega_n\tilde{Q}), \tag{24}$$

whose time-independence is evident, see (23) and (21*a*). Two other constants of motion are

$$\tilde{c}_n = \tilde{p}_n^2 + \omega_n^2\tilde{q}_n^2, \tag{25}$$

but they are not independent, since clearly $\tilde{c}_n = |\tilde{C}_n|^2$.

Let us emphasize that this property of *superintegrability* holds quite independently of the ratio α being *rational* or *irrational*. But let us also note that these five constants of motion are not very convenient to understand the consistency of this fact with the property of the 'embedded' dynamics yielded by the Hamiltonian h, see (15), to possess three constants of motion if α is a *rational* number but only two if α is *not* rational. Indeed the embedding procedure—as described above: entailing the restriction of the dynamics yielded by $\tilde{H}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q})$ to the invariant manifold characterized by $\tilde{H} = \tilde{P} = 0$ and the replacement of the canonical variable $\tilde{Q}$ with its time evolution on that manifold, $\tilde{Q}(t) = \tilde{Q}(0) + t$ (see (1*a*)), thereby causing the reduction from the six canonical variables featured by the Hamiltonian $\tilde{H}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q})$ to the four canonical variables featured by the Hamiltonian $h(\tilde{p}_1, \tilde{p}_2; \tilde{q}_1, \tilde{q}_2)$—clearly entails that the two complex constants of motion $\tilde{C}_n(\tilde{p}_n; \tilde{q}_n, \tilde{Q})$ are no more suitable to play the role of constants of motion for the dynamics yielded by the Hamiltonian $h(\tilde{p}_1, \tilde{p}_2; \tilde{q}_1, \tilde{q}_2)$. Indeed they—while being, of course, still constant when the canonical variables $\tilde{p}_1, \tilde{p}_2; \tilde{q}_1, \tilde{q}_2$ evolve according to the dynamics yielded by the Hamiltonian $h(\tilde{p}_1, \tilde{p}_2; \tilde{q}_1, \tilde{q}_2)$—feature now an explicit time-dependence in their definition, reading

$$\tilde{C}_n(\tilde{p}_n; \tilde{q}_n; t) = (\tilde{p}_n + \mathrm{i}\omega_n \tilde{q}_n) \exp\{-\mathrm{i}\omega_n[\tilde{Q}(0) + t]\}. \tag{26}$$

Since the main purpose of the example we are discussing is to display how this transition comes about—the transition from the *superintegrable* dynamics featured for arbitrary real α by the Hamiltonian $\tilde{H}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q})$ defined by (3) with (15), to the dynamics featured by the *embedded* Hamiltonian $h(\tilde{p}_1, \tilde{p}_2; \tilde{q}_1, \tilde{q}_2)$ being *superintegrable* if α is *rational* but being only *integrable* if α is *irrational*—let us now identify a more convenient set of five constants of motion associated with the Hamiltonian $\tilde{H}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q})$, assuming to begin with that $\tilde{\theta} \neq 0 \bmod(\pi)$. Three of these constants are of course again provided by the Hamiltonian $\tilde{H}$ itself and by the two constants $\tilde{c}_n$ (see (25)). To manufacture two other constants of motion we use the technique suggested by the proof provided in the appendix of [4], namely we take the time-average over the (closed) trajectory of our system, see (21), of the following, appropriately chosen, function of the canonical variables: $\exp[(\tilde{p}_1 + \mathrm{i}\omega_1\tilde{q}_1) + (\tilde{p}_2 - \mathrm{i}\omega_2\tilde{q}_2)]$. We thus get (using (23)) the following (complex) constant of motion:

$$\tilde{C}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q}) = \frac{\sin\tilde{\theta}}{2\pi} \int_0^{2\pi/\sin\tilde{\theta}} \mathrm{d}t \exp\{(\tilde{p}_1 + \mathrm{i}\omega_1\tilde{q}_1) \exp[\mathrm{i}\omega_1 \cot\tilde{\theta} \sin(t \sin\tilde{\theta})] + (\tilde{p}_2 - \mathrm{i}\omega_2\tilde{q}_2) \exp[-\mathrm{i}\omega_2 \cot\tilde{\theta} \sin(t \sin\tilde{\theta})]\}. \tag{27a}$$

Here of course the constant of motion $\tilde{\theta}$ is itself a function of the canonical variables, see for instance (22).

We now change the integration variable from t to $\tau = t \sin\tilde{\theta}$, power-expand the 'outer' exponential, use the binomial expansion for the powers thereby obtained, and end up with the following expression:

$$\tilde{C}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q}) = \sum_{m_1,m_2=0}^{\infty} \frac{(\tilde{p}_1 + \mathrm{i}\omega_1\tilde{q}_1)^{m_1} (\tilde{p}_2 - \mathrm{i}\omega_2\tilde{q}_2)^{m_2}}{m_1! m_2!} \times J_0[(m_1\omega_1 - m_2\omega_2) \cot\tilde{\theta}(\tilde{p}_1, \tilde{p}_2, \tilde{P}; \tilde{q}_1, \tilde{q}_2, \tilde{Q})], \tag{27b}$$

where we used the standard representation of the zeroth-order Bessel function,

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\tau \exp(\mathrm{i}x \sin\tau). \tag{28}$$

The real and imaginary parts of this quantity $\tilde{C}$ provide the two additional constants of motion; they are entire functions of the canonical coordinates, since the sum over the two indices m_1, m_2 clearly converges absolutely and uniformly in $\tilde{\theta}$ (recall that $|J_0(x)| \leqslant 1$ for all real values of x). And it is as well plain that these two constants are functionally independent among each other and with respect to the other three constants of motion, as identified above (the reader who doubts this is invited to verify this assertion by focussing on the behaviour of the system in the infinitesimal neighbourhood of the equilibrium configuration $\tilde{P} = 1/2$, $\tilde{Q} = 0$, $\tilde{p}_n = \tilde{q}_n = 0$). The *superintegrability* of the Hamiltonian $\tilde{H}(\vec{\tilde{p}}, \tilde{P}; \vec{\tilde{q}}, \tilde{Q})$ defined by (3) with (15) is thereby confirmed, at least for all motions *off* the invariant manifold M characterized by $\tilde{\theta} = 0 \mod(\pi)$.

The final issue to be investigated is the fate of these five constants of motion featured by this Hamiltonian $\tilde{H}$, when the motion is restricted to the invariant manifold M, in particular to what extent they are inherited by the dynamics associated with the embedded Hamiltonian h, see (15). First of all, of the three constants of motion $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{H}$ only the first two survive, since clearly $\tilde{H}$ vanishes identically, see (22).

Let us then look at the constant of motion $\tilde{C}$, see (27*b*), in the limit as $\tilde{\theta} \to 0 \mod(\pi)$, hence of course $\cot\tilde{\theta} \to \infty$.

Assume first that the two quantities ω_1 and ω_2 are *not* congruent—namely, their ratio is *not* a rational number. Then the argument of the Bessel function in the right-hand side of (27*b*) never vanishes, hence the divergence of $\cot\tilde{\theta}$ entails that the Bessel function vanishes (since $J_0(\infty) = 0$), and so does the entire sum (since every term of it vanishes, and the sum converges absolutely and uniformly in $\tilde{\theta}$). Hence $\tilde{C}$ disappears as $\tilde{\theta} \to 0 \mod(\pi)$, and we are only left with the two constants of motion $\tilde{c}_1$ and $\tilde{c}_2$, consistently with the fact that the Hamiltonian $h(\tilde{p}_1, \tilde{p}_2; \tilde{q}_1, \tilde{q}_2)$, see (15), is—in this case, with *noncongruent* ω_1 and ω_2—*integrable* but *not* superintegrable.

Assume instead that the two quantities ω_1 and ω_2 are *congruent*, say (consistently with (18)) $\omega_1 = r\omega$ and $\omega_2 = s\omega$ with r and s two positive integers and ω an arbitrary positive number. Then, whenever in the sum in the right-hand side of (27*b*) $m_1 = sm$ and $m_2 = rm$, the argument of the Bessel function vanishes, hence since $J_0(0) = 1$ we get

$$\tilde{C}(\tilde{p}_1, \tilde{p}_2; \tilde{q}_1, \tilde{q}_2) = \sum_{m=0}^{\infty} \frac{[(\tilde{p}_1 + \mathrm{i}\omega_1\tilde{q})^r (p_2 - \mathrm{i}\omega_2\tilde{q}_2)^s]^m}{(sm)!(rm)!}, \tag{29}$$

where we have omitted the rest of the sum (which clearly vanishes as $\tilde{\theta} \to 0 \mod(\pi)$, due to the same argument given above). This of course entails that the quantity $(\tilde{p}_1 + \mathrm{i}\omega_1\tilde{q})^r (p_2 - \mathrm{i}\omega_2\tilde{q}_2)^s$ is itself a constant of motion, recovering thereby the *superintegrability* of the dynamics yielded by $h(\tilde{p}_1, \tilde{p}_2; \tilde{q}_1, \tilde{q}_2)$ (as discussed above, compare indeed this constant of motion with (19)).

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